Modification of the Transition Rate in the Hydrogen Atom placed in Finite Space

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Abstract

When a hydrogen atom trapped in finite space formed by two parallel perfectly conducting plates, the transition rates can be modified. Life-time of the hydrogen $2P_{1/2}$-state is estimated to be shorter as much as $56.3ps$ with a separation distance of $b = 1.2\mu m$ between two plates. Although the modification is dependent on position of the atom, the life-time can be shorten or lengthen by adjusting the separation distance even for the case that the atom is placed at the center between two plates.

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§1. INTRODUCTION

A long time ago, Casimir[1] found that the zero-point energy appearing in quantization of the classical radiation field in free space could be observable by introducing two parallel perfectly conducting plates. He derived

$$\sum_{ljn} \frac{1}{2}\hbar\omega_{ljn} - \frac{bL^2}{(2\pi c)^3}\int_0^\infty \frac{1}{2}\hbar\omega d^3\omega = -\frac{\pi^2 L^2}{720 b^3}\hbar c, \tag{1}$$

where L^2 is the area of plates and b is the separation distance of two plates. The first and second terms on the left hand side denote the zero-point energies of radiation fields quantized in finite and free spaces, respectively. Notice the non-vanishing value on the right-hand side which is inversely proportional to cube of b. This result implies spatial dependence of the zero-point energy. On the other hand, it was also shown that the atomic energy levels could be modified when the atom was placed between two parallel plates [2,3,4]. Generally, the energy levels are expressed by real parts of the energy eigen-values while the life-times of energy levels, equivalently transition rates, are given by imaginary values of the complex energy eigen-values. Therefore, one may expect that the atomic transition rates will also be modified as long as the energy levels are shifted in finite space[5,6]. In this paper, we investigate the atomic transition rate in the framework similar to Casimir's calculation of the attractive force exerting between two plates [1].

§2. CALCULATION OF THE TRANSITION RATE

2.1 Case of free space

Let us start to briefly review the calculation of transition rate of the hydrogen atom in free space.

The time-dependent perturbation theory in quantum mechanics gives the first-order amplitude of spontaneous radiation as [7]

$$a_m^{(1)}(t) = \frac{1}{i\hbar}\int_0^t dt' \langle m|H_I(t')|l\rangle \exp\{i(E_m - E_l)t'/\hbar\}, \tag{2}$$

where the interaction Hamiltonian is expressed as

$$H_I(t') = H_I' \exp(\pm i\omega t'). \tag{3}$$

Then, the transition probability is

$$|a_m^{(1)}(t)|^2 = \frac{1}{\hbar^2}|\langle m|H_I'|l\rangle|^2|I_{ml}^{(\pm)}|^2, \tag{4}$$

where

$$I_{ml}^{(\pm)} = \int_0^t dt' \exp[i(\omega_0 \pm \omega)t'] \tag{5}$$

with $\hbar\omega_0 = E_m - E_l$. The negative(positive) sign in eq.(5) denotes the photon emission(absorption). Hereafter, we consider only the photon emission, i.e. negative sign in eq.(5). Carrying out the integration directly, we obtain

$$\begin{aligned}
|I_{ml}^{(-)}|^2 &= |\frac{\exp\{i(\omega_0-\omega)t\}-1}{i(\omega_0-\omega)}|^2 \\
&= 4\frac{\sin^2[\frac{t}{2}(\omega_0-\omega)]}{(\omega_o-\omega)^2}.
\end{aligned} \tag{6}$$

Then, the transition rate is given by [7,8]

$$\begin{aligned}
W_0 &= \frac{\partial}{\partial t}\int |a_m^{(1)}(t)|^2 \rho d(\hbar\omega) \\
&= \frac{2}{\hbar^2}\int |\langle m|H_I'|l\rangle|^2 \frac{\sin[t(\omega_0-\omega)]}{(\omega_0-\omega)}\frac{V\omega^2}{(2\pi)^3c^3}d\Omega d\omega.
\end{aligned} \tag{7}$$

If the dipole approximation is taken for H_I', eq.(7) can be given as

$$\begin{aligned}
W_0 &= \frac{2}{\hbar^2}\frac{\hbar}{2V}\int \sum_i |\langle m|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i}\boldsymbol{\epsilon}\cdot\boldsymbol{d}_i|l\rangle|^2 \frac{\sin[t(\omega_0-\omega)]}{(\omega_0-\omega)}\frac{V\omega^3}{(2\pi c)^3}d\Omega d\omega \\
&= \frac{e^2}{8\pi^3\hbar c^3}\int |\langle m|\boldsymbol{r}|l\rangle\cdot\boldsymbol{\epsilon}|^2 d\Omega \int_{-\infty}^{\infty}\frac{\omega^3\sin[t(\omega_0-\omega)]}{(\omega_0-\omega)}d\omega,
\end{aligned} \tag{7a}$$

where $\boldsymbol{\epsilon}$ is the polarization vector and $\boldsymbol{d} = e\boldsymbol{r}$ is a dipole moment. Considering that

$$\langle m|\boldsymbol{r}|l\rangle\cdot\boldsymbol{\epsilon} = \boldsymbol{r}_{ml}\cos\Theta, \tag{8}$$

and [7]

$$\int \cos^2 \Theta d\Omega = \frac{8\pi}{3}, \tag{9}$$

we find

$$\begin{aligned} W_0 &= \frac{e^2}{3\pi^2 \hbar c^3} |\boldsymbol{r}_{ml}|^2 \int_{-\infty}^{\infty} \frac{\omega^3 \sin[t(\omega_0 - \omega)]}{(\omega_0 - \omega)} d\omega \\ &= \left(\frac{e^2}{4\pi \hbar c} \right) \frac{4}{3} \frac{\omega_0^3}{c^2} |\boldsymbol{r}_{ml}|^2. \end{aligned} \tag{10}$$

The matrix element $\boldsymbol{r}_{ml}$ can be calculated with the hydrogen wave functions and the result for the $2P_{1/2} \to 1S_{1/2}$ transition is known as

$$|\boldsymbol{r}_{2P,1S}|^2 = \frac{1}{3} \left(a_0 \frac{16}{\sqrt{6}} \left(\frac{2}{3} \right)^4 \right)^2, \tag{11}$$

where a_0 is the Bohr radius. Since $\nu(2S_{1/2} \to 1S_{1/2}) = 2466061395.6(4.8) MHz$[9] and the Lamb shift yields $\nu(2S_{1/2} \to 2P_{1/2}) = 1057.8594 \pm 0.0019 MHz$ [10], we find $\nu(2P_{1/2} \to 1S_{1/2}) = \omega_0(2P_{1/2} \to 1S_{1/2})/2\pi = 2466060337.7 MHz$ and, then, the transition rate is

$$W_0(2P_{1/2} \to 1S_{1/2}) = 0.626288 \times 10^9 Hz, \tag{12}$$

which gives the life-time of the $2P_{1/2}$-state

$$\tau_0(2P_{1/2}) = \left[W_0(2P_{1/2} \to 1S_{1/2}) \right]^{-1} = 1.597 \times 10^{-9} s. \tag{13}$$

This is the result for the hydrogen atom in free space.

2.2 Coordinate system of dipole moment and polarization vector

Although the polarization vector, $\boldsymbol{\epsilon}$, is perpendicular to the wave vector, $\boldsymbol{k}$, of the radiation field, i.e. $\boldsymbol{\epsilon} \cdot \boldsymbol{k} = 0$, the dipole moment,$\boldsymbol{d}$, is generally not specified in any direction. It is better to clarify relations among these vectors.

First of all, the space-fixed coordinates are assigned by unit vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2,\ \boldsymbol{e}_3)$ and the wave vector of radiation field, $\boldsymbol{k}$, is chosen in the direction of a unit vector $\bar{\boldsymbol{e}}_3$. Here, unit vectors $(\bar{\boldsymbol{e}}_1,\ \bar{\boldsymbol{e}}_2,\ \bar{\boldsymbol{e}}_3)$ are orthogonal to each other. Then, we have

$$\bar{\boldsymbol{e}}_3 = \boldsymbol{e}_1 \sin\theta\cos\phi + \boldsymbol{e}_2 \sin\theta\sin\phi + \boldsymbol{e}_3\cos\theta \tag{15a}$$

where θ is the angle vetween $\boldsymbol{e}_3$ and $\bar{\boldsymbol{e}}_3$ and ϕ is the angle between $\boldsymbol{e}_1$ and a projection of $\bar{\boldsymbol{e}}_3$ on the $\boldsymbol{e}_1 - \boldsymbol{e}_2$ plane. Since $\bar{\boldsymbol{e}}_1$ is perpendicular to $\bar{\boldsymbol{e}}_3$, replacing θ by $\theta + \frac{\pi}{2}$ in eq.(15a), we find

$$\bar{\boldsymbol{e}}_1 = \boldsymbol{e}_1 \cos\theta\cos\phi + \boldsymbol{e}_2 \cos\theta\sin\phi - \boldsymbol{e}_3\sin\theta. \tag{15b}$$

Considering that $\bar{\boldsymbol{e}}_2$ is perpendicular to the $\bar{\boldsymbol{e}}_1 - \bar{\boldsymbol{e}}_3$ plane, we replace ϕ by $\phi + \frac{\pi}{2}$ and set $\theta = \frac{\pi}{2}$ in eq.(15a). Then, we can have

$$\bar{\boldsymbol{e}}_2 = -\boldsymbol{e}_1 \sin\phi + \boldsymbol{e}_2 \cos\phi. \tag{15c}$$

Polarization of the radiation field is perpendicular to $\boldsymbol{k}$, i.e. , $\boldsymbol{\epsilon}\cdot\boldsymbol{k} = 0$ and, then, it is expressed as

$$\boldsymbol{\epsilon}_i = \bar{\boldsymbol{e}}_i \quad (i = 1,\ 2). \tag{16}$$

If the dipole moment is along $\boldsymbol{e}_3$, i.e. $\boldsymbol{d} = d\boldsymbol{e}_3$, we find by eqs.(15b) and (15c)

$$\int_0^{2\pi} | < m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})|l > |^2 d\phi = \int_0^{2\pi} d_{ml}^2 \sin^2\theta d\phi = 2\pi d_{ml}^2(1-\cos^2\theta) = 2\pi d_{ml}^2\left(1 - \frac{k_z^2}{k^2}\right). \tag{17}$$

This result yields that given in eq.(4.9) of ref.11. On the other hand, for $\boldsymbol{d} = d\boldsymbol{e}_1$,

$$\begin{aligned}\int_0^{2\pi} | < m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})|l > |^2 d\phi &= d_{ml}^2 \int_0^{2\pi} [(\cos\theta\cos\phi)^2 + (\sin\phi)^2] d\phi \\ &= \pi d_{ml}^2(1+\cos^2\theta) = \pi d_{ml}^2\left(1 + \frac{k_z^2}{k^2}\right),\end{aligned} \tag{18}$$

which yields the result given in eq.(4.17) of ref.11. For $\boldsymbol{d} = d\boldsymbol{e}_2$, we obtain the same result as eq.(18),

$$\int_0^{2\pi} | < m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})|l > |^2 d\phi = \pi d_{ml}^2(1+\cos^2\theta) = \pi d_{ml}^2\left(1 + \frac{k_z^2}{k^2}\right). \tag{19}$$

Let us explore a general case that $\boldsymbol{d}$ is not polarized along a specific direction, i.e. $\boldsymbol{d}(\theta', \phi')$. When the normal mode of the electric field is expressed u_j $(j = x,\ y,\ z)$, we have

$$
\begin{aligned}
|<m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})u|l>|^2 \equiv \sum_i |<m|\epsilon_{ix}d_x u_x+\epsilon_{iy}d_y d_y+\epsilon_{iz}d_z u_z|l>|^2 \\
= d_{ml}^2\{|u_x\cos\theta\cos\phi\sin\theta'\cos\phi'+u_y\cos\theta\sin\phi\sin\theta'\sin\phi'-u_z\sin\theta\cos\theta'|^2 \\
+|-u_x\sin\phi\sin\theta'\cos\phi'+u_y\cos\phi\sin\theta'\sin\phi'|^2\}. \qquad (20)
\end{aligned}
$$

Integration over ϕ yields

$$
\begin{aligned}
\int_0^{2\pi}|<m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})u|l>|^2 d\phi = \pi d_{ml}^2\{|u_x|^2(1+\cos^2\theta)\sin^2\theta'\cos^2\phi' \\
+|u_y|^2(1+\cos^2\theta)\sin^2\theta'\sin^2\phi'+2|u_z|^2\sin^2\theta\cos^2\theta'\}, \quad (21)
\end{aligned}
$$

which becomes

$$
\int_0^{2\pi}|<m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})u|l>|^2 d\phi = \pi d_{ml}^2\{|u_{xy}|^2(1+\cos^2\theta)\sin^2\theta'+2|u_z|^2\sin^2\theta\cos^2\theta'\} \qquad (22)
$$

under an assumption of $u_x = u_y \equiv u_{xy}$. Notice that eq.(22) does not depend on ϕ' any more. When $u_{xy} = u_z = 1$ and $\theta' = 0$, i.e. $\boldsymbol{d}$ is along the z direction, eq.(22) leads to eq.(17), while with $u_{xy} = u_z = 1$ and $\theta' = \frac{\pi}{2}$, i.e. $\boldsymbol{d}$ is parallel to the x- or y-axis, eq.(22) leads to eq.(18). Furthermore, integration of eq.(22) over θ gives

$$
\int_0^{\pi}\int_0^{2\pi}|<m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})u|l>|^2\sin\theta d\theta d\phi = \pi d_{ml}^2\left\{\frac{8}{3}|u_{xy}|^2\sin^2\theta'+\frac{8}{3}|u_z|^2\cos^2\theta'\right\}, \qquad (23)
$$

which yields $\frac{8}{3}\pi d_{ml}^2$, when $u_{xy} = u_z = 1$. Actually, eqs.(9) and (10) were obtained by this manner.

2.3 Case of finite space

In finite space, the electromagnetic field has to be quantized to satisfy the boundary condition at the wall. And the energy-momentum tensor is constructed with this quantized field which is a solution of the Klein-Fok equation[12].

Since the electromagnetic field has a distinctive feature of transverse, the boundary condition is imposed at the wall such that the tangential components of the electric field vanish there. The simplest modification of photon propagator is to discretize the electromagnetic field modes in the direction perpendicular to the wall[13]. This photon propagator can be

proved[14] to be an approximation of the more reliable photon propagator derived by direct insertion of the δ - function to satisfy the boundary condition[15].

All these discussions lead to a result that the energy-momentum tensor gives the discretized energy in the vacuum state as

$$E = \int_0^b dx < 0|T_{00}|0 >= \frac{1}{2}\sum_n \omega_n \tag{24}$$

Therefore, the photon emitted spontaneously in finite space must have a discrete energy instead of the continuous one. If the hydrogen is placed between two parallel plates of size $L \times L$ seperated from each other by $b(b << L)$, z-component of the momentum of the radiation field which is perpendicular to the plate should be discretized and the integral over it has to be replaced by a summation. The result obtained by such a replacement is exactly identical with that derived by direct quantization of the electromagnetic field in finite space[11,16,17]. Compare eq.(7a) to eq.(4.7) of ref.11. Therefore, without repeating the field quantization in finite space, we shall start from eq.(7a) to obtain the transition rate in finite space. Nevertheless, our theory is rigorously based on the quantization given in ref.11.

In addition to such a replacement, we must change k_z to $\frac{\pi}{b}n$ $(n = 0,\ \pm1,\ \pm2,\ \cdots)$ and [11]

$$\frac{1}{\sqrt{V}}\exp(-i\boldsymbol{k}\cdot\boldsymbol{r}) \rightarrow \left\{ \begin{array}{rcl} \frac{1}{\sqrt{V}}u_z & = & \sqrt{\frac{2}{V}}\exp[-i(k_x x + k_y y)]\cos(k_z z) \\ \frac{1}{\sqrt{V}}u_{x,y} & = & \sqrt{\frac{2}{V}}\exp[-i(k_x x + k_y y)]i\sin(k_z z) \end{array} \right. \tag{25}$$

where a factor $\sqrt{2}$ comes from normalization.

Thus, the transition rate in finite space can be obtained from

$$W = \frac{2}{\hbar^2}\frac{\hbar}{2V}\int d\Omega \int_{-\infty}^{\infty} d\omega | < m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})\bar{u}(k_z R_z)|l > |^2 \frac{\omega^3 \sin[t(\omega_0 - \omega)]}{(\omega_0 - \omega)}\frac{V}{(2\pi c)^3}, \tag{26}$$

where $\bar{u}_x(k_z R_z) = \bar{u}_y(k_z R_z) = i\sin(k_z R_z)$ and $\bar{u}_z(k_z R_z) = \cos(k_z R_z)$. $R_z(0 < R_z \leq \frac{b}{2})$ is a distance between the atom and one of these plates. It should be reminded here that $\omega = ck = c(k_x^2 + k_y^2 + k_z^2)^{1/2}$ and the integral over k_z is replaced by a summation over n because of $k_z = \frac{\pi}{b}n$. Using the relation

$$\iiint_{-\infty}^{\infty} dk_x dk_y dk_z \frac{1}{k} = \int d\phi \int d\theta \sin\theta \int_0^\infty dk k = \frac{1}{c^2} \int d\phi \int d\theta \sin\theta \int_0^\infty d\omega\omega \tag{27}$$

and replacing the integral over k_z by a summation, we find the integral in eq.(26) as

$$\begin{aligned} I &\equiv \int d\Omega \int_{-\infty}^{\infty} d\omega | < m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})\bar{u}(k_z R_z)|l > |^2 \frac{\omega^3 \sin[t(\omega_0-\omega)]}{(\omega_0-\omega)} \\ &= \int d\Omega \int_0^\infty d\omega | < m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})\bar{u}(k_z R_z)|l > |^2 \left\{ \frac{\omega^3 \sin[t(\omega_0-\omega)]}{(\omega_0-\omega)} - \frac{\omega^3 \sin[t(\omega_0+\omega)]}{(\omega_0+\omega)} \right\}, \\ &\Rightarrow \frac{\pi c^4}{b} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y \sum_{n=-\infty}^{\infty} \left| \langle m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})\bar{u}(\frac{\pi n}{b} R_z)|l\rangle \right|^2 \\ &\quad \times \left\{ \frac{k \sin[t(\omega_0-ck)]}{(\omega_0-ck)} - \frac{k \sin[t(\omega_0+ck)]}{(\omega_0+ck)} \right\}, \end{aligned} \tag{28}$$

where $k = [k_x^2 + k_y^2 + (\frac{\pi}{b}n)^2]^{1/2}$. By transforming variables, k_x and k_y, into $(\frac{\pi}{L}i)$ and $(\frac{\pi}{L}j)$, equation (28) becomes

$$\begin{aligned} I = c^3 \left(\frac{\pi}{L}\right)^2 \left(\frac{\pi}{b}\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} didj \sum_{n=-\infty}^{\infty} \left| \langle m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})\bar{u}\left(\frac{\pi R_z}{b} n\right)|l\rangle \right|^2 \\ \times \left\{ \frac{B \sin[t\omega_0(1-B)]}{(1-B)} - \frac{B \sin[t\omega_0(1+B)]}{(1+B)} \right\}, \end{aligned} \tag{29}$$

where

$$B = \left[\left(\frac{c\pi}{\omega_0 L} i\right)^2 + \left(\frac{c\pi}{\omega_0 L} j\right)^2 + \left(\frac{c\pi}{\omega_0 b} n\right)^2 \right]^{1/2}. \tag{30}$$

Again changing variables as $i = (\frac{\omega_0 L}{c\pi})x$ and $j = (\frac{\omega_0 L}{c\pi})y$, we arrive at

$$\begin{aligned} I = \frac{c\pi\omega_0^2}{b} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy \sum_{n=-\infty}^{\infty} \left| \langle m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})\bar{u}(\frac{\pi R_z}{b} n)|l\rangle \right|^2 \\ \times \left\{ \frac{B_n \sin[t\omega_0(1-B_n)]}{(1-B_n)} - \frac{B_n \sin[t\omega_0(1+B_n)]}{(1+B_n)} \right\}, \end{aligned} \tag{31}$$

where

$$B_n = \left[x^2 + y^2 + \left(\frac{c\pi}{\omega_0 b} n\right)^2 \right]^{1/2}. \tag{32}$$

In order to evaluate the summation over n, it is convenient to make use of Poisson's summation formula on Fourier transforms,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{s=-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) \exp(2\pi isu) du. \tag{33}$$

Then, eq.(31) can be expressed as

$$
\begin{aligned}
I^{(\pm)} = &\frac{c\pi\omega_0^2}{b}\sum_{s=-\infty}^{\infty}\iiint_{-\infty}^{\infty} dxdydu \left|\langle m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})\bar{u}\left(\frac{\pi R_z}{b}u\right)|l\rangle\right|^2 \\
&\times\frac{\sqrt{x^2+y^2+\left(\frac{c\pi}{\omega_0 b}u\right)^2}\sin\left\{t\omega_0\left(1\pm\sqrt{x^2+y^2+\left(\frac{c\pi}{\omega_0 b}u\right)^2}\right)\right\}}{\left\{1\pm\sqrt{x^2+y^2+\left(\frac{c\pi}{\omega_0 b}u\right)^2}\right\}}\exp(2\pi isu) \\
= &\,\omega_0^3\sum_{s=-\infty}^{\infty}\iiint_{-\infty}^{\infty} dxdydz\left|\langle m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})\bar{u}\left(\frac{\omega_0 R_z}{c}z\right)|l\rangle\right|^2 \\
&\times\frac{\sqrt{x^2+y^2+z^2}\sin\left\{t\omega_0\left(1\pm\sqrt{x^2+y^2+z^2}\right)\right\}}{\left(1\pm\sqrt{x^2+y^2+z^2}\right)}\exp\left(i\frac{2\omega_0 bs}{c}z\right) \\
= &\,\omega_0^3\sum_{s=-\infty}^{\infty}\int d\Omega\int_0^{\infty} dr\left|\langle m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})\bar{u}\left(\frac{\omega_0 R_z}{c}r\cos\theta\right)|l\rangle\right|^2 \\
&\times\frac{r^3\sin\left\{t\omega_0\left(1\pm r\right)\right\}}{(1\pm r)}\exp\left(i\frac{2\omega_0 bs}{c}r\cos\theta\right).
\end{aligned}
\tag{34}
$$

Therefore, we obtain

$$
\begin{aligned}
I = &\,\omega_0^3\sum_{s=-\infty}^{\infty}\int d\Omega\int_0^{\infty} dr\left|\langle m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})\bar{u}\left(\frac{\omega_0 R_z}{c}r\cos\theta\right)|l\rangle\right|^2 \\
&\times\left\{\frac{r^3\sin\left\{t\omega_0\left(1-r\right)\right\}}{(1-r)}-\frac{r^3\sin\left\{t\omega_0\left(1+r\right)\right\}}{(1+r)}\right\}\exp\left(i\frac{2\omega_0 bs}{c}r\cos\theta\right) \\
= &\,\omega_0^3\sum_{s=-\infty}^{\infty}\int d\Omega\int_{-\infty}^{\infty} dr\left|\langle m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})\bar{u}\left(\frac{\omega_0 R_z}{c}r\cos\theta\right)|l\rangle\right|^2 \\
&\times\frac{r^3\sin\left\{t\omega_0\left(1-r\right)\right\}}{(1-r)}\exp\left(i\frac{2\omega_0 bs}{c}r\cos\theta\right) \\
= &\,\pi\omega_0^3\sum_{s=-\infty}^{\infty}\int d\Omega\left|\langle m|(\boldsymbol{\epsilon}\cdot\boldsymbol{d})\bar{u}\left(\frac{\omega_0 R_z}{c}\cos\theta\right)|l\rangle\right|^2\exp\left(i\frac{2\omega_0 bs}{c}\cos\theta\right),
\end{aligned}
\tag{35}
$$

where we used

$$
\frac{\sin[t\omega_0(1-r)]}{(1-r)}\simeq\pi\delta(1-r) \tag{36}
$$

for $\omega_0 t$ being very large, i.e. $\omega_0 = 1.5\times10^{16}s^{-1}$ and $t\simeq1.6\times10^{-9}s$ for the hydrogen $2P_{1/2}\to1S_{1/2}$ transition. By eq.(22), I can be given as

$$
\begin{aligned}
I = &\,\pi^2\omega_0^3 d_{ml}^2\int_0^{\pi} d\theta\sin\theta\left\{2\sin^2\left(\frac{\omega_0 R_z}{c}\cos\theta\right)\left(1+\cos^2\theta\right)\sin^2\theta'\right. \\
&\left.+4\cos^2\left(\frac{\omega_0 R_z}{c}\cos\theta\right)\left(1-\cos^2\theta\right)\cos^2\theta'\right\}\sum_{s=-\infty}^{\infty}\exp\left[i\frac{2\omega_0 bs}{c}\cos\theta\right].
\end{aligned}
\tag{37}
$$

By integrating over θ, eq.(37) is found in the form

$$I = \pi^2 \omega_0^3 d_{ml}^2 \sum_{s=-\infty}^{\infty} \Big\{ \sin^2\theta' [4 j_0(\xi_2) - 2 j_0(\xi_1+\xi_2) - 2 j_0(\xi_1-\xi_2)$$
$$- \frac{4}{\xi_2} j_1(\xi_2) + \frac{2}{(\xi_1+\xi_2)} j_1(\xi_1+\xi_2) + \frac{2}{(\xi_1-\xi_2)} j_1(\xi_1-\xi_2)]$$
$$+ \cos^2\theta' [\frac{8}{\xi_2} j_1(\xi_2) + \frac{4}{(\xi_1+\xi_2)} j_1(\xi_1+\xi_2) + \frac{4}{(\xi_1-\xi_2)} j_1(\xi_1-\xi_2)] \Big\}, \tag{38}$$

where $j(\xi)$ is the spherical Bessel function, $\xi_1 = \frac{2\omega_0 R_z}{c}$ and $\xi_2 = \frac{2\omega_0 b}{c} s$. Thus, the transition rate is given as

$$W = \frac{1}{8\pi^3 \hbar c^3} I = W_1 + W_2(b), \tag{39}$$

where

$$W_1 = W_0 \left\{ \sin^2\theta' \left[1 - \frac{3}{2} j_0(\xi_1) + \frac{3}{2\xi_1} j_1(\xi_1) \right] + \cos^2\theta' \left[1 + \frac{3}{\xi_1} j_1(\xi_1) \right] \right\}, \tag{40}$$

$$W_2(b) = W_0 \left\{ \sin^2\theta' \sum_{s=1}^{\infty} \left[3 j_0(\xi_2) - \frac{3}{2} j_0(\xi_1+\xi_2) - \frac{3}{2} j_0(\xi_1-\xi_2) \right.\right.$$
$$\left. - \frac{3}{\xi_2} j_1(\xi_2) + \frac{3}{2(\xi_1+\xi_2)} j_1(\xi_1+\xi_2) + \frac{3}{2(\xi_1-\xi_2)} j_1(\xi_1-\xi_2) \right]$$
$$\left. + \cos^2\theta' \sum_{s=1}^{\infty} \left[\frac{6}{\xi_2} j_1(\xi_2) + \frac{3}{(\xi_1+\xi_2)} j_1(\xi_1+\xi_2) + \frac{3}{(\xi_1-\xi_2)} j_1(\xi_1-\xi_2) \right] \right\}. \tag{41}$$

W_1 denotes the $s = 0$ term, and it is independent of b, while b dependence appears only in $W_2(b)$. Particularly, when $\boldsymbol{d}$ is polarized along the x or y direction, i.e. $\theta' = \frac{\pi}{2}$, W_1 leads to

$$W_{1x} = W_{1y} = W_0 \left[1 - \frac{3}{2} j_0(\xi_1) + \frac{3}{2\xi_1} j_1(\xi_1) \right]. \tag{42}$$

If $\boldsymbol{d}$ is along the z direction, i.e. $\theta' = 0$, it becomes

$$W_{1z} = W_0 \left[1 + \frac{3}{\xi_1} j_1(\xi_1) \right]. \tag{43}$$

These results are exactly the same as those given in refs. 11 and 16.

By defining

$$P_1 = \frac{3 j_1(\frac{2\omega_0 b}{c}(\frac{1}{2} - \frac{Z}{b}))}{(\frac{2\omega_0 b}{c}(\frac{1}{2} - \frac{Z}{b}))}, \tag{44}$$

$$P_2 = -\frac{3}{2}j_0(\frac{2\omega_0 b}{c}(\frac{1}{2} - \frac{Z}{b})) + \frac{1}{2}P_1, \tag{45}$$

$$Q_1(b) = \sum_{s=1}^{\infty}\left[3j_0\left(\frac{2\omega_0 b}{c}s\right) - \frac{3}{2}j_0\left(\frac{2\omega_0 b}{c}\left(s+\frac{1}{2}-\frac{Z}{b}\right)\right) - \frac{3}{2}j_0\left(\frac{2\omega_0 b}{c}\left(s-\frac{1}{2}+\frac{Z}{b}\right)\right)\right.$$
$$\left. - \frac{3j_1\left(\frac{2\omega_0 b}{c}s\right)}{\left(\frac{2\omega_0 b}{c}s\right)} + \frac{3j_1\left(\frac{2\omega_0 b}{c}\left(s+\frac{1}{2}-\frac{Z}{b}\right)\right)}{2\left(\frac{2\omega_0 b}{c}\right)\left(s+\frac{1}{2}-\frac{Z}{b}\right)} + \frac{3j_1\left(\frac{2\omega_0 b}{c}\left(s-\frac{1}{2}+\frac{Z}{b}\right)\right)}{2\left(\frac{2\omega_0 b}{c}\right)\left(s-\frac{1}{2}+\frac{Z}{b}\right)}\right], \tag{46}$$

$$Q_2(b) = \sum_{s=1}^{\infty}\left[\frac{6j_1\left(\frac{2\omega_0 b}{c}s\right)}{\left(\frac{2\omega_0 b}{c}s\right)} + \frac{3j_1\left(\frac{2\omega_0 b}{c}\left(s+\frac{1}{2}-\frac{Z}{b}\right)\right)}{\left(\frac{2\omega_0 b}{c}\right)\left(s+\frac{1}{2}-\frac{Z}{b}\right)} + \frac{3j_1\left(\frac{2\omega_0 b}{c}\left(s-\frac{1}{2}+\frac{Z}{b}\right)\right)}{\left(\frac{2\omega_0 b}{c}\right)\left(s-\frac{1}{2}+\frac{Z}{b}\right)}\right], \tag{47}$$

the transition rate can be written as

$$W = W_0[1 + \sin^2\theta'\{P_2 + Q_1(b)\} + \cos^2\theta'\{P_1 + Q_2(b)\}]. \tag{48}$$

Here $Z = \frac{b}{2} - R_z$ $(0 \leq Z < \frac{b}{2})$ is a distance of the atom in the z-direction from the center of finite space.

Since the dipole moment is not polarized in the specific direction in a practical experiment, we should take average values, $< \sin^2\theta' >=< \cos^2\theta' >= 1/2$, and, then

$$\bar{W} = W_0\{1 + \Delta_0 + \Delta(b)\} \tag{49}$$

with

$$\Delta_0 = \frac{1}{2}(P_1 + P_2), \tag{50}$$

$$\Delta(b) = \frac{1}{2}[Q_1(b) + Q_2(b)]. \tag{51}$$

For $b \to \infty$, the last term in eq.(49) vanishes but the second term, Δ_0, can survive as far as R_z is small, i.e. the atom is located closely to one of the walls. Accordingly, for the case of one wall, the modification can be given by Δ_0 - term.

§3. NUMERICAL RESULTS

With various values of b and Z, we can obtain numerical values of the shift of transition rate by eq.(49). Our results are shown in Table 1 for the hydrogen $2P_{1/2} \to 1S_{1/2}$ transition

with $Z = 0$. It can be seen that sign of the shift changes in accordance with b value. We define

$$\Delta W = \bar{W} - W_0 = W_0\{\Delta_0 + \Delta(b)\}, \tag{52}$$

values of which are also listed in Table 1.

For $Z = 0$ (notice $0 \leq Z < \frac{b}{2}$) and $b = 1.2\mu m$, we obtain $\Delta_0 = +0.00442, \Delta(b) = +0.0321$ and $\Delta W = +22.9$MHz, while $\Delta W = +38.2$MHz for $b = 0.7\mu m$. They are in a measureable range because the accuracy of the present technique is 4.8MHz[9].

The life-time is given by inverse of the transition rate,

$$\Delta\tau = \tau - \tau_0 \tag{53}$$

with $\tau_0 = 1.597$ns. Our results are listed in Table 1 and 2. Particularly, Table 2 shows dependence of the transition rate on position of the atom, when $b = 1.2\mu m$. At $Z = b/10$ which is very close to the center, $\Delta_0 + \Delta(b)$ yields $+0.036$, i.e. $\Delta W = +22.8$MHz. This is corresponding to $\Delta\tau = -56.1ps$.

§4. CONCLUSION

In this paper, we have calculated effects of the finite space on the transition rate and the life-time of the hydrogen atom.

We have derived explicitly b dependence of the transition rate, $\Delta(b)$, which was not evaluated in the previous calculations [11,17]. The result shows the characteristic oscillation of the transition rate in accordance with b values. This property depends very much on value of the frequency ω_0. The magnitude of $\Delta(b)$ varies with value of ω_0. b-dependence is around $2 \sim 5\%$ for a separation distance of μm order. However, it is much enhanced for smaller value of ω_0. For example, with $\omega_0 = 1.2 \times 10^{11}$Hz, we find that contribution of the b-dependence term, $\Delta(b)$, is 33% for $b = 1cm$.

As is seen in Table 2, variation of the transition rate and life-time are very sensitive to position of the atom. Since it is quite hard to set up the atom at an accurate position,

measurement of the absolute magnitudes of ΔW and $\Delta\tau$ may be very difficult. However, observation of these shifts seems to be definitely possible.

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TABLES

Table 1. Shifts of the transition rate and the life-time for $Z = 0$. $\omega_0 = 1.5 \times 10^{16}$Hz.

$b(\mu m)$	$Q_1(b) \times 10^2$	$Q_2(b) \times 10^3$	$\Delta_0 \times 10^2$	$\Delta(b) \times 10^2$	ΔW(MHz)	$\Delta\tau$(ps)
0.5	-0.1532	-9.127	0.038	-0.5330	-3.10	7.94
0.6	2.053	0.9236	2.423	1.073	21.90	-53.95
0.7	9.865	1.556	1.081	5.010	38.15	-91.69
0.8	-5.800	1.196	-1.301	-2.840	-25.93	68.99
0.9	-0.5994	-0.09178	-1.474	-0.3043	-11.14	28.91
1.0	-0.02956	-1.843	0.306	-0.1070	1.25	-3.17
1.1	1.475	0.3780	1.360	0.7565	13.26	-33.10
1.2	6.375	0.5344	0.442	3.214	22.90	-56.33
1.3	-3.090	0.4390	-0.923	-1.523	-15.32	40.04
1.4	-0.2568	-0.1551	-0.858	-0.1362	-6.23	16.04
1.5	0.003605	-0.6429	0.351	-0.03034	2.01	-4.45
1.6	1.281	0.2144	0.935	0.6511	9.93	-24.93
1.7	4.958	0.2672	0.186	2.493	16.78	-41.67
1.8	-1.903	0.2196	-0.732	-0.9030	-10.24	26.54
1.9	-0.1193	-0.1560	-0.557	-0.06743	-3.91	10.03
2.0	0.02723	-0.2726	0.360	-0.000014	2.25	-5.73

Table 2. Dependence of the transition rate and the life-time on Z for $b = 1.2\mu m$. $\omega_0 = 1.5 \times 10^{16}$ Hz.

Z/b	$\Delta_0 \times 10^2$	$\Delta(b) \times 10^2$	ΔW(MHz)	$\Delta\tau$(ps)
0	0.442	3.214	22.90	-56.33
1/20	0.841	2.811	22.87	-56.27
1/19	0.459	3.202	22.93	-56.40
1/18	-0.0233	3.441	21.40	-52.78
1/17	-0.564	3.435	17.98	-44.57
1/16	-1.075	3.088	12.61	-31.51
1/15	-1.410	2.349	5.88	-14.86
1/14	-1.374	1.289	-0.532	1.36
1/13	-0.791	0.2083	-3.65	9.36
1/12	0.306	-0.2960	0.06	-0.16
1/11	1.375	0.4584	11.48	-28.75
1/10	1.265	2.376	22.80	-56.10
1/9	-0.618	3.347	17.09	-42.42
1/8	-1.478	0.9838	-3.10	7.93
1/7	1.528	0.4776	12.56	-31.40
1/6	-1.303	2.974	10.47	-26.25
1/5	2.088	1.486	22.38	-55.11

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